

Quasi-isometric embedding of the fundamental group of an orthogonal graph-manifold into a product of metric trees

Alexander Smirnov*

Abstract

In every dimension $n \geq 3$ we introduce a class of orthogonal graph-manifolds and prove that the fundamental group of any orthogonal graph-manifold quasi-isometrically embeds into a product of n trees. As a consequence, we obtain that asymptotic and linearly-controlled asymptotic dimensions of such group are equal to n .

1 Introduction

We introduce a class \mathcal{O} of orthogonally glued higher-dimensional graph-manifolds (that we call throughout this paper *orthogonal graph-manifolds*; see section 2.2 for the definition). Using the ideas of the paper [7], we generalize the results of that paper to the case of the class \mathcal{O} .

Theorem 1. *For every n -dimensional orthogonal graph-manifold its fundamental group supplied with an arbitrary word metric admits a quasi-isometric embedding into a product of n metric trees. As a consequence, asymptotic and linearly-controlled asymptotic dimensions of such group are equal to n .*

In the paper [7] this result was obtained in the 3-dimensional case for every graph-manifold in the sense of the definition in section 2.1. In fact, according to the paper [8], the fundamental group of any 3-dimensional graph-manifold is quasi-isometric to the fundamental group of some flip-manifold, which is precisely an orthogonal graph-manifold in the dimension 3. Also note that the inequality $\text{asdim } \pi_1(M) \leq n$ for the fundamental group of an orthogonal graph-manifold M follows from the result obtained in the Bell – Dranishnikov [3].

*Supported by RFFI Grant 11-01-00302-a

2 Preliminaries

2.1 Graph-manifolds

Definition. A higher-dimensional graph-manifold is a closed, orientable, n -dimensional, $n \geq 3$, manifold M that is glued from a finite number of blocks M_v , $M = \bigcup_{v \in V} M_v$. These should satisfy the following conditions (1)–(3).

- (1) Each block M_v is a trivial T^{n-2} -bundle over a compact, orientable surface Φ_v with boundary (the surface should be different from the disk and the annulus), where T^{n-2} is a $(n-2)$ -dimensional torus;
- (2) the manifold M is glued from blocks M_v , $v \in V$, by diffeomorphisms between boundary components (the case of gluing boundary components of the same block is not excluded);
- (3) gluing diffeomorphisms do not identify the homotopy classes of the fiber tori.

For brevity, we use the term “graph-manifold” instead of the term “higher-dimensional graph-manifold”.

Let G be a graph dual to the decomposition of M into blocks. The set of blocks of the graph-manifold coincides with the vertex set $V = V(G)$ of the graph G . The set of (non-oriented) edges $E = E(G)$ of G consists of pairs of glued components of blocks. We denote the set of the oriented edges of G by W .

For more information about the graph-manifolds see [4].

2.2 Orthogonal graph-manifolds

In this section we define a class of graph-manifolds that admit an orthogonally glued metric of a special form. For brevity, we will call them orthogonal graph-manifolds.

Fix a graph G and for each vertex $v \in V(G)$ consider a surface Φ_v of nonnegative Euler characteristic with $|\partial_v|$ boundary components, where ∂_v is the set of all edges adjacent to the vertex v . Moreover, we assume that there is a bijection between the set of boundary components of the surface Φ_v and the set of all oriented edges adjacent to v . For the block M_v corresponding to a vertex v we fix a trivialization $M_v = \Phi_v \times S^1 \times \dots \times S^1$, where Φ_v is the base surface, i.e. we fix simultaneously a trivialization $M_v = \Phi_v \times T^{n-2}$ of M_v and a trivialization $T^{n-2} = S^1 \times \dots \times S^1$ of the fiber torus. For each block M_v , we fix a coordinate system (x, x_1, \dots, x_{n-2}) compatible with this decomposition, where $x \in \Phi_v$ and $x_i \in [0, 1)$ for each $1 \leq i \leq n-2$. For each oriented edge w adjacent to the vertex v , we define the coordinate system (x_0) , $x_0 \in [0, 1)$ on the corresponding component of the boundary $\partial\Phi_v$ of the surface Φ_v . It defines the coordinate system (x_0, \dots, x_{n-2}) on the boundary

torus T_w of the block M_v . Similarly, for the edge $-w$ inverse to the edge w on the boundary torus T_{-w} , we define the coordinate system (x'_0, \dots, x'_{n-2}) . For each oriented edge w , we consider a permutation \mathfrak{s}_w of a well-ordered $(n-1)$ -element set (x_0, \dots, x_{n-2}) such that $\mathfrak{s}_w(x_0) \neq x_0$. Furthermore, we assume that for mutually inverse edges w and $-w$ the permutations \mathfrak{s}_w and \mathfrak{s}_{-w} are inverse ($\mathfrak{s}_{-w} \circ \mathfrak{s}_w = \text{id}$). We define the gluing map $\eta_w: T_w \rightarrow T_{-w}$ by $\eta_w((x_0, \dots, x_{n-2})) = (\mathfrak{s}_w(x_0), \dots, \mathfrak{s}_w(x_{n-2}))$. Note that this map is a well-defined gluing, as permutations \mathfrak{s}_w and \mathfrak{s}_{-w} are selected to be mutually inverse. Also, the map η_w does not identify the homotopy classes of fiber tori.

Definition. The above described graph-manifold is called *an orthogonal graph-manifold*.

Remark 1. As mentioned above, in the case $n = 3$ the class of all orthogonal graph-manifolds coincides with the class of all flip graph-manifolds considered in [8].

2.3 Metric trees

A *tripod* in a geodesic metric space X is a union of three geodesic segments $xt \cup yt \cup zt$ which have only one common point t . A geodesic metric space X is called a *metric tree* if each triangle in it is a tripod (possibly degenerate).

2.4 Finitely generated groups

Let G be a finitely generated group and $S \subset G$ a finite symmetric generating set for G ($S^{-1} = S$). Recall that a *word metric* on the group G (with respect to S) is the left-invariant metric defined by the norm $\|\cdot\|_S$, where for each $g \in G$ its norm $\|g\|_S$ is the smallest number of elements of S whose product is g . It is known that all such metrics for the group G are bi-Lipschitz equivalent (see [2]). In this paper we will consider only finitely generated groups with a word metric.

2.5 Quasi-isometric maps

A map $f: X \rightarrow Y$ is said to be *quasi-isometric* if there exist $\lambda \geq 1, C \geq 0$ such that

$$\frac{1}{\lambda}|xy| - C \leq |f(x)f(y)| \leq \lambda|xy| + C$$

for each $x, y \in X$. Metric spaces X and Y are called *quasi-isometric* if there is a quasi-isometric map $f: X \rightarrow Y$ such that $f(X)$ is a net in Y . In this case, f is called a *quasi-isometry*.

2.6 The metric on the universal cover

Let us recall the famous Milnor–Švarc Lemma.

Lemma 1. *Let Y be a compact length space and let X be the universal cover of Y considered with the metric lifted from Y . Then X is quasi-isometric to the fundamental group $\pi_1(Y)$ of the space Y considered with an arbitrary word metric.*

It follows from this lemma that to prove Theorem 1 it is sufficient to construct a quasi-isometric embedding of the universal cover into a product of n trees.

2.6.1 Metrics of non-positive curvature

Define a metric on the orthogonal graph-manifold M as follows: for each edge $e \in E(G)$ take a flat metric on its corresponding torus T_e such that any base circle of the coordinate system described above has length 1, and any two of these circles are perpendicular.

In particular, for each vertex $v \in V(G)$ there is a metric on the boundary surface Φ_v in which every boundary component has length 1. This metric can be extended to a metric of nonpositive curvature on the surface Φ_v so that its boundary is geodesic. Therefore, the metric from the boundary tori extends to the metric on the block M_v , which is locally a product metric (in general, a metric on the block may not be a product metric and it can have nontrivial holonomy along some loops on the base Φ_v).

Further we consider only those metrics on orthogonal graph-manifolds. If we lift the above metric in the universal cover \tilde{M} , it follows from the Reshetnyak gluing theorem (see [2]) that the obtained metric space is nonpositively curved (or Hadamard) space.

We fix an orthogonal graph-manifold M with the metric described above.

2.6.2 The standard hyperbolic surface with boundary H_0

Consider the hyperbolic plane \mathbb{H}_κ^2 having a curvature $-\kappa$ ($\kappa > 0$) such that the side of a rectangular equilateral hexagon θ in the plane \mathbb{H}_κ^2 has length 1. Let ρ be the distance between the middle points of sides, which have a common adjacent side, δ the diameter of θ . We mark each second side of θ (so we have marked three sides) and consider a set H_0 defined as follows. Take the subgroup G_θ of the isometry group of \mathbb{H}_κ^2 generated by reflections in (three) marked sides of θ and let H_0 be the orbit of θ with respect to G_θ . Then H_0 is a convex subset in \mathbb{H}_κ^2 divided into hexagons that are isometric to θ . Furthermore, the boundary of H_0 has infinitely many connected components each of which is a geodesic \mathbb{H}_κ^2 . The graph T_{bin} dual to the decomposition of H_0 into hexagons is the standard binary tree whose vertices all have degree three. Any metric space isometric to H_0 will be

called a θ -tree. Given a vertex p of T_{bin} , we denote by θ_p the respective hexagon in H_0 .

Remark 2. In what follows, we will consider T_{bin} as the metric space with a metric such that the length of each edge is equal to 2ρ . Then these metric spaces are metric trees. We will denote the set of vertices in T_{bin} by $V(T_{bin})$.

2.6.3 Standard metrics and bi-Lipschitz homeomorphisms between bases

Consider a simplicial tree with the degree 3 of each vertex, and the length 1 of each edge. We replace each edge by the rectangle $1 \times 1/3^{100}$ and each vertex by the equilateral Euclidean triangle with each side equal to $1/3^{100}$. Then we glue them in a natural way.

Definition. The obtained metric space is called a *fattened tree* or a *standard surface* and denoted by X_0 .

Remark 3. Note that the standard surface X_0 is bi-Lipschitz homeomorphic to the θ -tree H_0 . So we fix an arbitrary bi-Lipschitz homeomorphism $h_0: H_0 \rightarrow X_0$.

Definition. The *standard block* is defined to be a metric product of the θ -tree H_0 and $n - 2$ copies of the Euclidean line \mathbb{R} , $B = H_0 \times \mathbb{R} \times \dots \times \mathbb{R}$.

The partition of the θ -tree by hexagons induces a partition of each boundary component of the θ -tree by unit segments. For each boundary component such a partition is called a *grid* on this component. For each Euclidean factor \mathbb{R} of the standard block, an arbitrary partition by unit segments is called a *grid* on this factor. Finally for each boundary hyperplane σ of the standard block a partition by unit cubes induced by grids on each factor is called a *grid* on this hyperplane.

Recall the following theorem.

Theorem 2 ([1], Theorem 1.2). *Let X_0 be as above with a chosen boundary component $\partial_0 X_0$. Then there exists $K > 0$ and a function $\psi: \mathbb{R} \rightarrow \mathbb{R}$ such that for any $K_0 > 0$ and any K_0 -bi-Lipschitz homeomorphism P_0 from $\partial_0 X_0$ to a boundary component $\partial_1 X_0$, P_0 extends to a $\psi(K_0)$ -bi-Lipschitz homeomorphism $P: X_0 \rightarrow X_0$ which is K -bi-Lipschitz on every other boundary component.*

Corollary 1. *Let $\tilde{\Phi}_v$ be the universal cover of the surface Φ_v supplied with the metric described in sect. 2.6.1 with a chosen boundary component $\partial_0 \tilde{\Phi}_v$. Then there exists $K > 0$ and a function $\psi: \mathbb{R} \rightarrow \mathbb{R}$ such that for any $K_0 > 0$ and any K_0 -bi-Lipschitz homeomorphism P_0 from $\partial_0 \tilde{\Phi}_v$ to a boundary component $\partial_0 H_0$, P_0 extends to a $\psi(K_0)$ -bi-Lipschitz homeomorphism $P: \tilde{\Phi}_v \rightarrow H_0$ which is K -bi-Lipschitz on every other boundary component.*

Proof. Rename the K_0 , K and ψ from Theorem 2 to the \bar{K}_0 , \bar{K} and $\bar{\psi}$ respectively. Note that there is a bi-Lipschitz homeomorphism $\psi_v: X_0 \rightarrow \tilde{\Phi}_v$.

$$\begin{array}{ccc} X_0 & \longrightarrow & X_0 \\ \psi_v \downarrow & & \uparrow h_0 \\ \tilde{\Phi}_v & \longrightarrow & H_0 \end{array}$$

Let ψ_v be M_1 -bi-Lipschitz and h_0 be M_2 -bi-Lipschitz. We set $M := \max\{M_1, M_2\}$. Consider $P_0: \partial_0 \tilde{\Phi}_v \rightarrow \partial_0 H_0$. Denote the boundary component $h_0(\partial_0 H_0)$ by $\partial_1 X_0$. Then if $\partial_0 X_0 = \psi_v^{-1}(\partial_0 \tilde{\Phi}_v)$ the map

$$h_0 \circ P_0 \circ \psi_v: \partial_0 X_0 \rightarrow \partial_1 X_0$$

is $M^2 \cdot K_0$ -bi-Lipschitz homeomorphism. By Theorem 2 it extends to a $\bar{\psi}(M^2 \cdot K_0)$ -bi-Lipschitz homeomorphism that is \bar{K} -bi-Lipschitz on each remaining boundary component of the space X_0 . Therefore, the homeomorphism $P = h_0^{-1} \circ \bar{P} \circ \psi_v^{-1}$ is $\psi(K_0) = M^2 \bar{\psi}(M^2 \cdot K_0)$ -bi-Lipschitz. Moreover, on every other boundary component it is $K = M^2 \cdot \bar{K}$ -bi-Lipschitz. Therefore, for $\psi(x) = \bar{\psi}(M^2 \cdot x)$, $x \in \mathbb{R}$ and $K = M^2 \cdot \bar{K}$ Corollary 1 is proved. \square

Remark 4. Since the graph G is finite, we can assume that the number K and the function ψ are independent of $v \in V(G)$.

2.6.4 The special metric on the universal cover

In this section we inductively construct for each orthogonal graph-manifold M a special metric on its universal covering \widetilde{M} so that \widetilde{M} with such a metric is quasi-isometric to the fundamental group $\pi_1(M)$ of the graph-manifold M . Afterwards it will be sufficient to construct a quasi-isometric embedding of \widetilde{M} into a product of n trees.

The decomposition of M into blocks lifts to a decomposition of \widetilde{M} into universal cover blocks, see [4]. We denote the tree dual to this decomposition by T_0 . Note that the degree of every vertex of T_0 is infinite. On T_0 , we consider an intrinsic metric with length 1 edges. Choose a vertex $o \in V(T_0)$ in the tree T_0 and call it *the root*. For each vertex $v \in V(T_0)$, we define its rank $r(v)$ as the distance to o . In particular, $r(o) = 0$. For each vertex $v \in V(T_0)$, we denote by \widetilde{M}_v the corresponding block of the space \widetilde{M} . This block is isometric to the product $\tilde{\Phi}_v \times \mathbb{R}^{n-2}$. Recall that on each boundary hyperplane of the block \widetilde{M}_v we have fixed a coordinate system. We call the axes of this system *the selected axes*.

By induction on the rank of vertices of the tree T_0 , we construct on the space \widetilde{M} a metric of special type, which is bi-Lipschitz homeomorphic to the metric lifted from the graph-manifold M .

Base: Let $\psi_o : \widetilde{\Phi}_o \rightarrow H_0$ be a bi-Lipschitz homeomorphism. Consider an isometric copy of the standard block B_o and consider the map $\psi'_o : \widetilde{M}_o \rightarrow B_o$ which is a direct product of the map $\psi_o : \widetilde{\Phi}_o \rightarrow H_0$ and the identity map $\text{id} : \mathbb{R}^{n-2} \rightarrow \mathbb{R}^{n-2}$.

Inductive step: Suppose that for all vertices $v \in V(T_0)$ such that $r(v) \leq m$ we built a bi-Lipschitz homeomorphism $\psi'_v : \widetilde{M}_v \rightarrow B_v$ where B_v is an isometric copy of the standard block. Consider a vertex $u \in V(T_0)$ such that $r(u) = m + 1$. There is a unique vertex $v \in V(T_0)$ adjacent to it such that $r(v) = m$. Consider the blocks \widetilde{M}_v and \widetilde{M}_u of the universal cover \widetilde{M} . Denote the covering map by $\text{pr} : \widetilde{M} \rightarrow M$. Recall that the gluing of the blocks $\text{pr}(\widetilde{M}_v)$ and $\text{pr}(\widetilde{M}_u)$ is obtained by the permutation \mathfrak{s} of the coordinate system on the torus T_e . Consider an isometric copy of the standard block. Denote it by B_u and glue it to the block B_v by the permutation \mathfrak{s}^{-1} of the coordinates along the corresponding hyperplanes thus matching the grid on them. By the induction, the map $\psi'_v : \widetilde{M}_v \rightarrow B_v$ is the direct product of the map $\widetilde{\Phi}_v \rightarrow H_0$ and $n - 1$ maps $\mathbb{R} \rightarrow \mathbb{R}$. Moreover, the restriction of each of these maps to the intersection with the common boundary hyperplane of the blocks \widetilde{M}_v and \widetilde{M}_u is a K -bi-Lipschitz homeomorphism onto its image. It follows from the orthogonality of the gluing that these restrictions induce a K -bi-Lipschitz homeomorphism $\partial_1 \psi_u : \partial_0 \widetilde{\Phi}_u \rightarrow \partial_0 H_0$ from the boundary component $\partial_0 \widetilde{\psi}_u$ of the surface $\widetilde{\Phi}_u$ adjacent to the block \widetilde{M}_v to the boundary component $\partial_0 H_0$ of the θ -tree adjacent to the block B_v . Also, these restrictions induce a collection of K -bi-Lipschitz homeomorphisms $\partial_i : \mathbb{R} \rightarrow \mathbb{R}$, each of which maps the corresponding \mathbb{R} -factor of the decomposition $\widetilde{M}_u = \widetilde{\Phi}_u \times \mathbb{R} \times \dots \times \mathbb{R}$ to the corresponding \mathbb{R} -factor of the decomposition $B_u = H_0 \times \mathbb{R} \times \dots \times \mathbb{R}$.

By Corollary 1, the homeomorphism $\partial_1 \psi_u$ extends to a $\psi(K)$ -bi-Lipschitz homeomorphism $\psi_u : \widetilde{\Phi}_u \rightarrow H_0$, which is K -bi-Lipschitz on every other boundary component. We define the homeomorphism ψ'_u as the direct product of the homeomorphism ψ_u and $n - 3$ homeomorphisms ∂_i ($i = 2, \dots, n - 2$).

Let us construct a map $\psi_M : \widetilde{M} \rightarrow X$, where X is a metric space obtained by gluing blocks described above. Namely, if the point x lies in the block \widetilde{M}_v for some vertex $v \in V(T_0)$ we define $\psi_M(x) := \psi'_v(x)$. The map ψ_M is well defined, since the maps ψ'_v are compatible with each other.

Proposition 1. *The map constructed above is a bi-Lipschitz homeomorphism.*

Proof. Let $C = \max\{K, \psi(K)\}$. It follows from the construction that for each vertex $u \in V(T_0)$ the map ψ'_u is C -bi-Lipschitz. Suppose that $x \in \widetilde{M}_v$ for some vertex $v \in V(T)$ and $y \in \widetilde{M}_u$ for some vertex $u \in V(T)$. Denote $x' := \psi_M(x)$ and $y' := \psi_M(y)$.

Let γ be a geodesic between vertices v and u in the tree T . Denote its

consecutive edges by e_1, \dots, e_k . Note that a geodesic $xy \subset \widetilde{M}$ consecutively intersects hyperplanes $\sigma_1, \dots, \sigma_k$ in the space \widetilde{M} that correspond to these edges. Similarly, a geodesic $x'y' \subset X$ consistently intersects hyperplanes $\sigma'_1, \dots, \sigma'_k$ in the space X . Moreover, $\sigma'_i = \psi_M(\sigma_i)$. Let z_i be an intersection point of the geodesic xy and the hyperplane σ_i . (We assume that $z_0 = x$, $z_{k+1} = y$.)

Let $z'_i = \psi_M(z_i)$. Since for each vertex v the restriction of the map ψ_M on the block \widetilde{M}_v is C -bi-Lipschitz and the points z_i and z_{i+1} ($i = 0, \dots, k$) lie in the same block, we have $|z'_i z'_{i+1}| \leq C|z_i z_{i+1}|$. Combining all these inequalities, we find that $\sum_{i=0}^k |z'_i z'_{i+1}| \leq C|xy|$. On the other hand, by the triangle inequality we have $|x'y'| \leq \sum_{i=0}^k |z'_i z'_{i+1}|$. This implies that $|x'y'| \leq C|xy|$. Similarly, we have $|xy| \leq C|x'y'|$. \square

Thus, we define a metric of special type on the universal cover of the orthogonal graph-manifold. Such a metric has nonpositive curvature in the sense of Alexandrov. Moreover, for every vertex $v \in V(T_0)$ the corresponding block \widetilde{M}_v is isometric to the direct product of the θ -tree H_0 and $n-2$ factors \mathbb{R} .

Let us introduce some technical notations that will be needed later. For each vertex v and the corresponding block of \widetilde{M}_v , denote by X_v a copy of the corresponding θ -tree. Moreover, denote a copy of the tree T_{bin} naturally (isometrically) embedded in the surface X_v , considered with the above described metric, by T_v . Let G'_θ be the isometry group of the θ -tree. Note that for each vertex v there exists 2δ -Lipschitz retraction $r_v: X_v \rightarrow T_v$ equivariant under the action of G'_θ .

Recall that the block \widetilde{M}_v is a product $X_v \times \mathbb{R} \times \dots \times \mathbb{R}$. Denote the projections to the corresponding factors by p_v^1, \dots, p_v^{n-1} .

For each point $x \in \widetilde{M}_v$ consider the map given by

$$\pi_x(y) := (y, p_v^2(x), \dots, p_v^{n-1}(x)) \text{ for every point } y \in X_v.$$

We call an orthogonal graph-manifold *irreducible* if its universal cover \widetilde{M} is not a product of a Euclidean space and universal cover of the orthogonal graph-manifold of lower dimension. It suffices to prove Theorem 1 for the irreducible case.

3 Trees T_c and maps to them

3.1 Construction of trees

Let $\gamma = w_1 \dots w_k$ be an oriented path in the tree T_0 . Denote by \mathfrak{s}_γ the permutation $\mathfrak{s}_{w_k} \circ \dots \circ \mathfrak{s}_{w_1}$ of well-ordered $(n-1)$ -element set.

Suppose that vertices $u, v \in T_0$ are connected by two oriented paths γ_1 and γ_2 . Note that $\mathfrak{s}_{\gamma_1} = \mathfrak{s}_{\gamma_2}$, therefore, we can define the permutation \mathfrak{s}_{uv} as the permutation \mathfrak{s}_γ along any path γ between u and v . Furthermore, $\mathfrak{s}_{v_1 v_3} = \mathfrak{s}_{v_2 v_3} \circ \mathfrak{s}_{v_1 v_2}$ and $\mathfrak{s}_{uv} = \mathfrak{s}_{vu}^{-1}$.

We define a relation \sim on the set of vertices of T_0 by $u \sim v$ if and only if the permutation \mathfrak{s}_{uv} fixes the smallest element. It is easy to check that the relation \sim is an equivalence relation.

Let us prove that the relation \sim divides the set $V(T_0)$ into not more than $n - 1$ equivalence classes. Indeed, if it fails, then we can choose n pairwise non-equivalent vertices v_1, \dots, v_n . Then, for some different $2 \leq i, j \leq n$, we have $\mathfrak{s}_{v_1 v_i}(x_0) = \mathfrak{s}_{v_1 v_j}(x_0)$, where x_0 is the smallest element. Then, since $\mathfrak{s}_{v_i v_j}(x_0) = \mathfrak{s}_{v_1 v_j} \circ \mathfrak{s}_{v_i v_1}(x_0) = x_0$, $v_i \sim v_j$. This is a contradiction.

Fix a vertex u in the tree T_0 . Since the manifold M is irreducible, the set of permutations $\{\mathfrak{s}_{vu} \mid u, v \in V(T_0)\}$ is transitive. That is, for each element x of a well-ordered $(n - 1)$ -element set, there is a vertex $v \in V(T_0)$ that $\mathfrak{s}_{vu}(x_0) = x$. It follows that there are at least $n - 1$ different equivalence classes. Hence there are exactly $n - 1$.

Denote the set of all these classes by \mathcal{C} . We have shown that $|\mathcal{C}| = n - 1$. Given $c \in \mathcal{C}$, note that if $u, v \in c$ then for any vertex $v' \in V(T_0)$ we have $\mathfrak{s}_{uv'}(x_0) = \mathfrak{s}_{v'v}(x_0)$.

Fix a vertex $v \in V(T_0)$ and an equivalence class $c \in \mathcal{C}$. We construct a tree $T_{v,c}$ as follows. If the vertex v belongs to the class c then we set $T_{v,c} := T_v$, see the end of section 2. Otherwise, we set $T_{v,c} := \mathbb{R}$.

Also, for each vertex $v \in V(T_0)$ we construct a map $r_{v,c}: \widetilde{M}_v \rightarrow T_{v,c}$. If the vertex v belongs to c then we set $r_{v,c} := r_v \circ p_v^1$. Otherwise, if the vertex v belongs to some class $c' \neq c$, then we set $r_{v,c} = p_v^k$, where $k = \mathfrak{s}_{uv}(x_0)$, $u \in c$, and x_0 is the smallest element.

For each class $c \in \mathcal{C}$, we construct a tree of T_c as follows. For each pair of adjacent vertices $u, v \in V(T_0)$, we say that a point $x \in T_{u,c}$ and a point $y \in T_{v,c}$ are \sim_c -equivalent if there exists a point $z \in \widetilde{M}_u \cap \widetilde{M}_v$ such that $x = r_{u,c}(z) = r_{v,c}(z) = y$.

This relation is well defined. Indeed, for every point $x \in T_{u,c}$ the preimage $r_{u,c}^{-1}(x) \cap \widetilde{M}_u \cap \widetilde{M}_v$ is an $(n - 2)$ -dimensional subspace orthogonal to the coordinate $\mathfrak{s}_{v'u}(x_0)$, where the vertex v' belongs to c . Similarly, for each point $y \in T_{v,c}$ the preimage $r_{v,c}^{-1}(y) \cap \widetilde{M}_u \cap \widetilde{M}_v$ is an $(n - 2)$ -dimensional subspace orthogonal to the coordinate $\mathfrak{s}_{v'v}(x_0)$. But by the definition of coordinates $\mathfrak{s}_{v'u}(x_0)$ and $\mathfrak{s}_{v'v}(x_0)$, any two such subspaces are either disjoint or coincide. This implies immediately the following lemma.

Lemma 2. *Let $u, v \in V(T_0)$ be a pair of adjacent vertices and $c \in \mathcal{C}$ be an equivalence class. If the points $x, x' \in T_{u,c}$ and $y \in T_{v,c}$ are such that $x \sim_c y$ and $x' \sim_c y$, then $x = x'$.*

Extend the relation \sim_c by transitivity. This means that we set $x \sim_c y$

if and only if there exists a chain $x = x_0, \dots, x_l = y$ that $x_i \sim_c x_{i+1}$ and $x_i \in T_{v_i, c}$ for each $0 \leq i \leq l-1$, and the vertices v_i and v_{i+1} are adjacent in the tree T_0 . From Lemma 2, it follows that the relation \sim_c is an equivalence relation.

Lemma 3. *Let $c \in \mathcal{C}$ be an equivalence class. Fix any pair of vertices $u, v \in c$, and consider points $x, y \in T_{u, c}$ and points $x', y' \in T_{v, c}$ such that $x \sim_c x'$ and $y \sim_c y'$. Then $|xy| = |x'y'|$.*

Proof. In fact, let $u = v_0, \dots, v_l = v$ be consecutive vertices of the geodesic between vertices u and v in the tree T_0 . Note that it suffices to consider the case when the vertices v_1, \dots, v_{l-1} are not in the class c . For each $0 \leq i \leq l-1$ denote the common hyperplane of the blocks \widetilde{M}_{v_i} and $\widetilde{M}_{v_{i+1}}$ by σ_i . Let ∂X_u be the common boundary component of the θ -tree X_u corresponding to the hyperplane σ_0 . Well as, let ∂X_v be the common boundary component of the θ -tree X_v corresponding to the hyperplane σ_{l-1} . By the construction of the metric on the universal cover \widetilde{M} for each interval I_0 of the grid on the boundary component ∂X_u , there are segments I_1, \dots, I_{l-2} of the grids on the corresponding \mathbb{R} -factors and the segment I_{l-1} of the grid on the boundary component ∂X_v such that for each $1 \leq i \leq l-1$ we have $p_{v_i}^j(I_{i-1}) = p_{v_i}^j(I_i)$, where $j = \mathfrak{s}_{uv_i}(x_0)$. Note that it is sufficient to prove the Lemma for arbitrary points $x, y \in r_{u, c}(I_0) \subset T_{u, c}$. But for such points we have $x', y' \in r_{v, c}(I_{l-1})$, hence, by the equivariance of retractions r_u and r_v the required equality $|xy| = |x'y'|$ is satisfied. \square

Fix $c \in \mathcal{C}$. Define the space T_c as a factor $\{\bigsqcup T_{v, c} \mid v \in V(T_0)\} / \sim_c$. From Lemma 3, it follows that the resulting space is a metric tree. Moreover, for each vertex $v \in c$ the natural embedding $proj_{v, c}: T_{v, c} \rightarrow T_c$ is isometric.

Thus, for each $c \in \mathcal{C}$ we constructed a tree T_c , which is naturally divided into blocks $T_{v, c}$.

3.2 Maps to trees

The remainder of this paper, we consider the product of $|\mathcal{C}| + 1 = n$ of constructed trees $T_0 \times \prod_{c \in \mathcal{C}} T_c$ as a metric space with the sum metric. It means that the distance between two points $x, y \in T_0 \times \prod_{c \in \mathcal{C}} T_c$ defined as the sum of the distances between their projections in the trees T_0, T_c , for each $c \in \mathcal{C}$.

$$|xy| = |x_0 y_0|_{T_0} + \sum_{c \in \mathcal{C}} |x_c y_c|_{T_c}.$$

We define a map $\varphi_c: \widetilde{M} \rightarrow T_c$ by the formula:

$$\varphi_c(x) := (proj_{v, c} \circ r_{v, c})(x),$$

where $x \in \widetilde{M}_v$ and $proj_{v,c}$ is the natural embedding of the tree $T_{v,c}$ in the tree T_c . It follows from the definition of the maps $r_{v,c}$ and from the construction of the tree T_c that the map φ_c is well defined on the intersection $\widetilde{M}_u \cap \widetilde{M}_v$ of each pair of adjacent blocks \widetilde{M}_u and \widetilde{M}_v .

From the definition of the map $r_{v,c}$, we have that φ_c is 2δ -Lipschitz. To prove this, it suffices to show that

$$|\varphi_c(x)\varphi_c(y)| \leq 2\delta|xy|,$$

where x and y belong to the same block \widetilde{M}_v . This fact follows from the definition of φ_c .

Define a map $\varphi_0: \widetilde{M} \rightarrow T_0$ as follows. If $x \in \widetilde{M}_v$ and for any other vertex $u \in V(T_0)$, $x \notin \widetilde{M}_u$ set $\varphi_0(x) = v$. Otherwise if $x \in \widetilde{M}_u \cap \widetilde{M}_v$ and $r(u) < r(v)$ set $\varphi_0(x) = u$. It is clear that $|\varphi_0(x)\varphi_0(y)| \leq |xy| + 1$.

Define a map $\varphi: \widetilde{M} \rightarrow T_0 \times \prod_{c \in \mathcal{C}} T_c$ by the equality $\varphi := \varphi_0 \times \prod_{c \in \mathcal{C}} \varphi_c$.

Then

$$|\varphi(x)\varphi(y)| = |\varphi_0(x)\varphi_0(y)| + \sum_{c \in \mathcal{C}} |\varphi_c(x)\varphi_c(y)| \leq (2\delta(n-1) + 1)|xy| + 1.$$

4 Special curves

For a curve γ in a metric space X by $|\gamma|$ denote its length. For further proof we need the following lemma.

Lemma 4. *Let $x, y \in \widetilde{M}$ be a pair of points. Then there exists a curve $\gamma \subset \widetilde{M}$ between them such that $|\gamma| \leq (2\delta + 1)|\varphi(x)\varphi(y)| + 2\delta$.*

Proof. Consider vertices $u, v \in V(T_0)$ such that $x \in \widetilde{M}_v$, $y \in \widetilde{M}_u$. We prove the lemma by induction on the length of the path $|uv|_{T_0}$.

Base: There exists a vertex $v \in V(T_0)$ that $x, y \in \widetilde{M}_v$. In this case, the geodesic xy does the job. Indeed,

$$|xy| = \sqrt{|p_1^v(x)p_1^v(y)|^2 + \dots + |p_{n-1}^v(x)p_{n-1}^v(y)|^2},$$

that does not exceed

$$2\delta + \sum_{c \in \mathcal{C}} |\varphi_c(x)\varphi_c(y)| \leq (2\delta + 1)|\varphi(x)\varphi(y)| + 2\delta.$$

Inductive step: Fix vertices $u, v \in V(T_0)$. Let η_0 be a geodesic between u and v in the tree T_0 . Choose vertices $u', v' \in \eta_0$ that vertices u and u' as well as vertices v and v' are adjacent. Note that either $r(u) > r(u')$ or $r(v) > r(v')$. Without loss of generality, assume that $r(v) > r(v')$. We can also assume that the point x does not belong to the block $\widetilde{M}_{v'}$. In this

case, $\varphi_0(x) = v$ and for any point $x' \in \widetilde{M}_v \cap \widetilde{M}_{v'}$ we have $\varphi_0(x') = v'$. It follows that $|\varphi_0(x)\varphi_0(x')| = 1$.

Assume the vertex v belongs to the class $c \in \mathcal{C}$. Consider a geodesic η between $\varphi(x)$ and $\varphi(y)$. Denote its projection to the tree T_c by η_c . Note that the curve η_c is a geodesic between $\varphi_c(x)$ and $\varphi_c(y)$. Divide the curve η_c into two parts $\eta_c^1 = \eta_c \cap T_{v,c}$ and $\eta_c^2 = \eta_c \setminus \eta_c^1$. Let z be the end of the curve η_c^1 different from $\varphi_c(x)$.

Recall that $\pi_x: X_v \rightarrow \widetilde{M}_v$ is a horizontal embedding such that the image contains the point x . Further, we assume that the tree $T_{v,c}$ is naturally embedded in the θ -tree X_v . Define a map π'_x as the composition of such embedding and the map π_x . Denote the image of the curve η_c^1 under the map π'_x by γ_c . We set $x_0 := \pi'_x(\varphi_c(x))$ and $x_1 := \pi'_x(z)$. Then $|xx_0| < \delta$ and there exists a point $x_2 \in \widetilde{M}_v \cap \widetilde{M}_{v'}$ such that $x_2 \in \pi_x(X_v)$, $\varphi_c(x_2) = z$ and for any $c \neq c' \in \mathcal{C}$ we have $\varphi_{c'}(x_2) = \varphi_{c'}(x)$. Note that $|x_1x_2| < \delta$ and

$$|\varphi_{c'}(x)\varphi_{c'}(x_2)| + |\varphi_{c'}(x_2)\varphi_{c'}(y)| = |\varphi_{c'}(x)\varphi_{c'}(y)|.$$

On the other hand, since $\varphi_c(x_2) = z$,

$$|\varphi_c(x)\varphi_c(x_2)| + |\varphi_c(x_2)\varphi_c(y)| = |\varphi_c(x)z| + |z\varphi_c(y)| = |\varphi_c(x)\varphi_c(y)|.$$

Finally, we note that the point $\varphi_0(x_2)$ belongs to the geodesic between $\varphi_0(x)$ and $\varphi_0(y)$ in the tree T_0 . It follows that

$$|\varphi_0(x)\varphi_0(x_2)| + |\varphi_0(x_2)\varphi_0(y)| = |\varphi_0(x)\varphi_0(y)|.$$

It means that

$$|\varphi(x)\varphi(x_2)| + |\varphi(x_2)\varphi(y)| = |\varphi(x)\varphi(y)|,$$

and $x_2 \in \widetilde{M}_{v'}$. By induction, for the points x_2 and y , there exists a curve γ' between x_2 and y with

$$|\gamma'| \leq (2\delta + 1)|\varphi(x_2)\varphi(y)| + 2\delta.$$

Consider the curve γ which is the union of the geodesic xx_0 , the curve γ_c , the geodesic x_1x_2 and the curve γ' . We have

$$|\gamma| = |xx_0| + |\gamma_c| + |x_1x_2| + |\gamma'| \leq 2\delta + |\gamma_c| + |\gamma'| \leq 2\delta + |\varphi_c(x)\varphi_c(x_2)| + |\gamma'|,$$

which by induction does not exceed

$$2\delta + |\varphi(x)\varphi(x_2)| + (2\delta + 1)|\varphi(x_2)\varphi(y)| + 2\delta.$$

We have shown that $|\varphi_0(x)\varphi_0(x_2)| = 1$, therefore,

$$2\delta + |\varphi(x)\varphi(x_2)| = 2\delta|\varphi_0(x)\varphi_0(x_2)| + |\varphi(x)\varphi(x_2)| \leq (2\delta + 1)|\varphi(x)\varphi(x_2)|.$$

So

$$\begin{aligned} |\gamma| &\leq 2\delta + |\varphi(x)\varphi(x_2)| + (2\delta + 1)|\varphi(x_2)\varphi(y)| + 2\delta \leq \\ &\leq (2\delta + 1)(|\varphi(x)\varphi(x_2)| + |\varphi(x_2)\varphi(y)|) + 2\delta, \end{aligned}$$

hence $|\gamma| \leq (2\delta + 1)(|\varphi(x)\varphi(y)|) + 2\delta$. \square

Corollary 2. *For any points $x, y \in \widetilde{M}$ the inequality $|xy| \leq (2\delta + 1)|\varphi(x)\varphi(y)| + 2\delta$ holds.*

Applying the above inequalities, we obtain

$$|xy|/(2\delta + 1) - 2\delta/(2\delta + 1) \leq |\varphi(x)\varphi(y)| \leq (2\delta(n - 1) + 1)|xy| + 1,$$

therefore,

$$1/C|xy| - 1 \leq |\varphi(x)\varphi(y)| \leq C|xy| + 1,$$

where $C = \max\{2\delta + 1, 2\delta(n - 1) + 1\}$. This completes the proof of Theorem 1.

5 Asymptotic dimensions

5.1 Definitions

Recall some basic definitions and notations. Let X be a metric space. We denote by $|xy|$ the distance between $x, y \in X$ and $d(U, V) := \inf\{|uv| \mid u \in U, v \in V\}$ is the distance between $U, V \subset X$.

We say that a family \mathcal{U} of subsets of X is a *covering* if for each point $x \in X$ there is a subset $U \in \mathcal{U}$ such that $x \in U$. A family \mathcal{U} of sets is *disjoint* if each two sets $U, V \in \mathcal{U}$ are disjoint. The union $\mathcal{U} = \cup\{\mathcal{U}^\alpha \mid \alpha \in \mathcal{A}\}$ of disjoint families \mathcal{U}^α is said to be *n-colored*, where $n = |\mathcal{A}|$ is the cardinality of \mathcal{A} .

Also, recall that a family \mathcal{U} is *D-bounded*, if the diameter of every $U \in \mathcal{U}$ does not exceed D , $\text{diam } U \leq D$. A n -colored family of sets \mathcal{U} is *r-disjoint*, if for every color $\alpha \in \mathcal{A}$ and each two sets $U, V \in \mathcal{U}^\alpha$ we have $d(U, V) \geq r$.

The linearly-controlled asymptotic dimension is a version of the Gromov's asymptotic dimension, asdim .

Definition. (Gromov [6]) The *asymptotic dimension* of a metric space X , $\text{asdim } X$, is the least integer number n such that for each sufficiently large real R there exists a $(n + 1)$ -colored, R -disjoint, D -bounded covering of the space X , where the number $D > 0$ is independent of R .

Definition. (Roe [9]) The *linearly-controlled asymptotic dimension* of a metric space X , $\ell\text{-asdim } X$, is the least integer number n such that for each sufficiently large real R there exists an $(n + 1)$ -colored, R -disjoint, CR -bounded covering of the space X , where the number $C > 0$ is independent of R .

It follows from the definition that $\text{asdim } X \leq \ell\text{-asdim } X$ for any metric space X .

In the next section we show that the fundamental group of orthogonal graph-manifold satisfies $n \leq \text{asdim } \pi_1(M) \leq \ell\text{-asdim } \pi_1(M) \leq n$.

5.2 Upper and lower bounds

Recall some properties of the above dimensions.

Let X and Y be metric spaces. If X is quasi-isometric to Y then $\text{asdim } X = \text{asdim } Y$ and $\ell\text{-asdim } X = \ell\text{-asdim } Y$. If $X \subset Y$ then $\ell\text{-asdim } X \leq \ell\text{-asdim } Y$. Also, $\ell\text{-asdim } X \times Y \leq \ell\text{-asdim } X + \ell\text{-asdim } Y$. Let T be a metric tree, then $\ell\text{-asdim } T \leq 1$. It follows from the above properties, that $\text{asdim } \pi_1(M) = \text{asdim } \widetilde{M} \leq \ell\text{-asdim } \widetilde{M} \leq \ell\text{-asdim}(T_0 \times \prod_{c \in \mathcal{C}} T_c) \leq n$. On the other hand, the space \widetilde{M} is an Hadamard manifold, and hence, see [5, Theorem 10.1.1], $\text{asdim } \widetilde{M} \geq n$.

References

- [1] BEHRSTOCK J. A. AND NEUMANN W. D.: *Quasi-isometric classification of graph manifold groups*, Duke Math. J., **Volume 141, Number 2** (2008), 217-240.
- [2] BURAGO D., BURAGO Y., IVANOV S.: *A Course in Metric Geometry*, AMS Bookstore, 2001.
- [3] BELL G. AND DRANISHNIKOV A.: *On Asymptotic Dimension of Groups Acting on Trees*, Geometriae Dedicata, **Volume 103, Number 1** (2004), 89-101.
- [4] BUYALO S. V., KOBEL'SKII V.L.: *Generalized graphmanifolds of non-positive curvature*, St. Petersburg Math. J. 11 (2000), 251–268.
- [5] BUYALO S. V., SCHROEDER V.: *Elements of asymptotic geometry*, EMS monographs in mathematics, European Mathematical Society, 2007.
- [6] GROMOV M.: *Asymptotic invariants of infinite groups*, London Mathematical Society Lecture Note Series, **Volume 182** (1993), 1-295.
- [7] HUME D., SISTO A.: *Embedding universal covers of graph manifolds in products of trees*, preprint arXiv:math.GT/1112.0263
- [8] KAPOVICH M. AND LEEB B.: *3-manifold groups and nonpositive curvature*, Geometric Analysis and Functional Analysis, **Volume 8** (1998), 841-852.

- [9] ROE J.: *Lectures on Coarse Geometry*, University Lecture Series, **Volume 31**, AMS, 2003.